2 Fabry-Perot resonator

2.1 Perfectly reflective surfaces, $R = 1$

Consider the case of a plane wave bouncing back and forth between two perfectly reflective surfaces ($R_a = R_b = 1$). The electric field between the surfaces will be

$$\begin{aligned} E &= E_o e^{-i(\omega t - kz)} + r E_o e^{-i(\omega t + kz)} \\ &= E_0 e^{-i\omega t}\left(e^{-ikz} + r e^{ikz}\right) \end{aligned}$$

where r is the field reflection coefficient.[1]

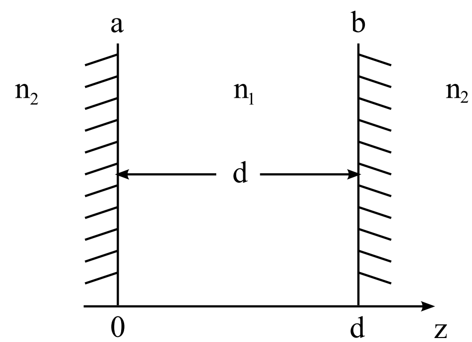


Figure 1: Two perfectly reflective surfaces.

The electric field E has to be zero at the interfaces: $E(z=0) = E(z=d) = 0$.

$$\begin{array}{llllll} z = 0: & 1 + r = 0 & \Rightarrow & r = -1 & \\ z = d: & e^{-ikd} = e^{ikd} & \Rightarrow & kd = q\pi, & q = 1, 2, \ldots \end{array}$$

Therefore, we have that

$$\nu = q\frac{c}{2d}$$

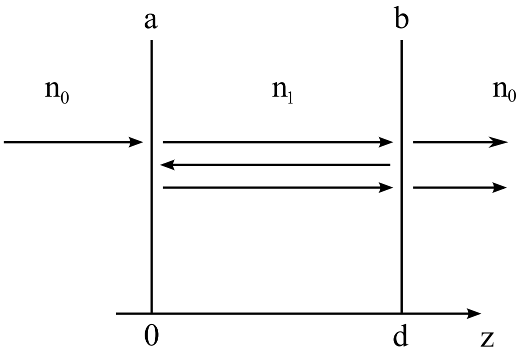
so only discrete frequencies (or modes) are allowed, with a spacing of

$$\Delta\nu = \frac{c}{2d}$$

This is the so-called free spectral range and characterizes the shift in frequency neccessary to shift the fringe system from the resonator by exactly one fringe.

[1]The reflectance R is related to the field reflection coefficient r by $R = |r|^2$

2.2 Etalon, $R < 1$



Fresnel:
$$r_{01} = -r_{10}$$
$$n_1 t_{01} = n_0 t_{10}$$
$$R_{01} = R_{10} = R$$
$$T_{01} = \tfrac{n_1}{n_0} t_{01}^2 = t_{01} t_{10}$$

Now, allowing for some transmission through the interfaces we can calculate the field reflection coefficient of the etalon.

$$\begin{aligned}
r_{etalon} = \frac{E_{reflected}}{E_0} &= r_{01} + t_{01} r_{10} t_{10} e^{2ikd} + t_{01} r_{10}^3 t_{10} e^{4ikd} + \cdots \\
&= r_{01}\left[1 - t_{01} t_{10} e^{2ikd}\left(1 + r_{10}^2 e^{2ikd} + \cdots\right)\right] \\
&= r_{01}\left[1 - \frac{Te^{2ikd}}{1 - Re^{2ikd}}\right] \\
&= r_{01}\frac{1 - e^{2ikd}}{1 - Re^{2ikd}}
\end{aligned}$$

The reflectance of the etalon R_{etalon} is then given by

$$\begin{aligned}
R_{etalon} = |r_{etalon}|^2 &= \frac{R\left[(1-\cos 2kd)^2 + \sin^2 2kd\right]}{(1 - R\cos 2kd)^2 + R^2 \sin^2 2kd} \\
&= \frac{2R(1 - \cos 2kd)}{1 - 2R\cos 2kd + R^2} \\
&= \frac{4R \sin^2 kd}{(1-R)^2 + 4R\sin^2 kd}
\end{aligned}$$

And the transmission by

$$\begin{aligned}
T_{etalon} = 1 - R_{etalon} &= \frac{(1-R)^2}{(1-R)^2 + 4R\sin^2 kd} \\
&= \frac{1}{1 + \frac{4R\sin^2 kd}{(1-R)^2}}
\end{aligned}$$

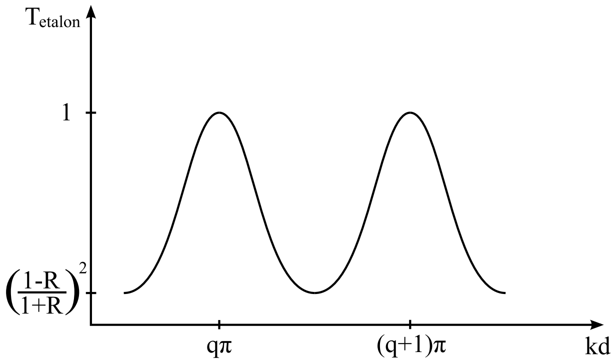


Figure 2: The maximum and minimum transmission is $T_{max} = 1$ and $T_{min} = \left(\frac{1-R}{1+R}\right)^2$

Since T_{etalon} doesn't go down to zero we can only talk about the half width of the peaks when the following condition is satisfied: $R \gg \frac{\sqrt{2}-1}{\sqrt{2}+1} \simeq 17\%$. When $T_{etalon} = 1/2$, then

$$(1-R)^2 = 4R\sin^2 kd \qquad \text{or} \qquad \sin kd = \frac{1-R}{2\sqrt{R}} \simeq \phi_{1/2}$$

The finesse F is a measure of the sharpness of the interference fringes:

$$F = \frac{\pi}{2\phi_{1/2}} = \frac{\pi\sqrt{R}}{1-R}$$

The spacing between the resonances is determined by the condition $kd = q\pi$, $q = 1, 2, \ldots$ as before, so

$$\nu = q\frac{c}{2n_1 d} \qquad \text{and} \qquad \Delta\nu = \frac{c}{2n_1 d}$$

The peaks have a finite width of $2\phi_{1/2}$ (FWHM) since energy is lost from the resonator ($R < 1$).

3 Beam Tracing and Mirror Resonators

Ray tracing is an practical implementation of paraxial ray analysis in optical system design. It's foundation is the paraxial approximation of Snell, that is $\sin\theta \simeq \theta$. The **thin lens equation** $1/f = 1/a + 1/b$ is only valid in that case.

3.1 Ray transfer matrices

A paraxial ray is characterized by its distance r from the symmetry axis and the angle r' it makes with the axis. By representing the beam with the vector

$$\vec{r} = \begin{bmatrix} r \\ r' \end{bmatrix}$$

we will be able to build a system of linear equations to trace the beam through a optical system.

3.1.1 Lens

We assume that the lens is thin. We use the subscript 1 for the incident beam and 2 for the outgoing beam. The lens changes the slope of the beam, but not the distance from the axis, that is

$$r_2 = r_1 \qquad \text{and} \qquad r_1' = \frac{r_1}{a}$$

the thin lens equation gives us

$$\frac{1}{b} = \frac{1}{f} - \frac{1}{a} = \frac{1}{f} - \frac{r_1'}{r_1}$$

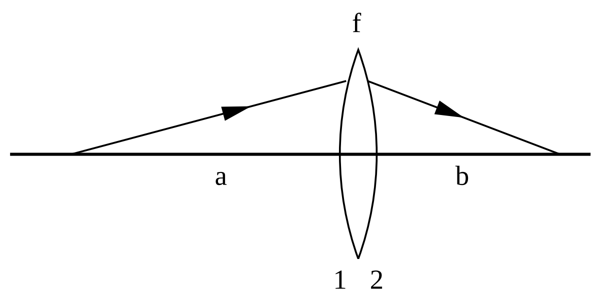


Figure 3: Ray path through a thin lens.

and therefore

$$r_2' = -\frac{r_2}{b} = r_1' - \frac{r_1}{f}$$

We can now write the relations between the incident and the outgoing beams using matrices:

$$\begin{bmatrix} r_2 \\ r_2' \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}_f \begin{bmatrix} r_1 \\ r_1' \end{bmatrix} \qquad \begin{array}{rcl} r_2 &=& Ar_1 + Br_1' \\ r_2' &=& Cr_1 + Dr_1' \end{array}$$

and from the equations above we can see that $A = 1$, $B = 0$, $C = -1/f$ and $D = 1$. And finally we get the **transfer matrix**

$$\mathbf{M}_f = \begin{bmatrix} A & B \\ C & D \end{bmatrix}_f = \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix}$$

and we can write $\vec{r}_2 = \mathbf{M}_f \vec{r}_1$

3.1.2 Ray traveling a distance d

It is easy to see that a ray traveling through a uniform optical medium of length d can be described as

$$\begin{aligned} r_2 &= r_1 + d r_1' \\ r_2' &= r_1' \end{aligned}$$

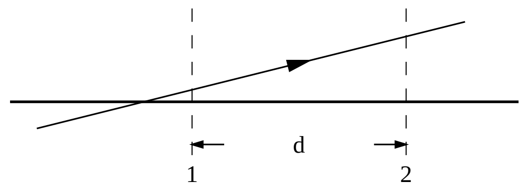


Figure 4: Ray traveling a distance d.

Therefore the transfer matrix can be written as

$$\mathbf{M}_d = \begin{bmatrix} A & B \\ C & D \end{bmatrix}_d = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$$

3.1.3 The Propagation of Rays in Mirror Resonators

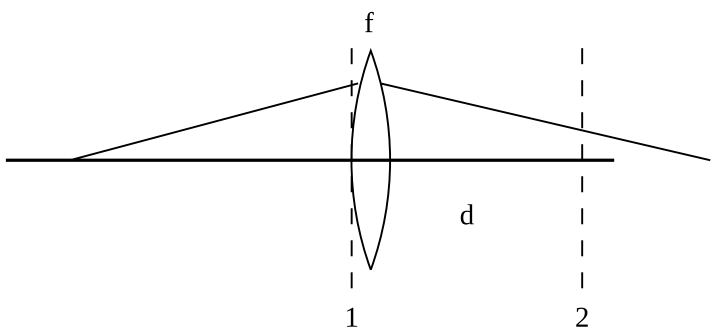


Figure 5: Ray going through a thin lens and traveling distance d.

By combining the results for the tranfer matrices $\mathbf{M}_f$ and $\mathbf{M}_d$ and get

$$\vec{r}_2 = \mathbf{M}_d \mathbf{M}_f \vec{r}_1 = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_1' \end{bmatrix} = \underbrace{\begin{bmatrix} 1 - d/f & d \\ -1/f & 1 \end{bmatrix}}_{\mathbf{M}_t} \vec{r}_1$$

The curved mirror resonator shown in figure 6(a) is equivalent to the periodic lens sequence shown in fig 6(b). We can calculate the total transfer matrix

$$\mathbf{M} = \mathbf{M}_d \mathbf{M}_{f_2} \mathbf{M}_d \mathbf{M}_{f_1} = \begin{bmatrix} (1 - d/f_2)(1 - d/f_1) - d/f_1 & (2 - d/f_2)d \\ -(1 - d/f_1)/f_2 - 1/f_1 & 1 - d/f_2 \end{bmatrix}$$

$\mathbf{M}^n \vec{r}$ then describes the transmission on the ray through n lenses (reflections). It can be shown that transfer matricies have determinant of unity. For such matrices we can use Sylvester's theorem

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^n = \frac{1}{\sin\theta} \begin{bmatrix} A \sin n\theta - \sin(n-1)\theta & B \sin n\theta \\ C \sin n\theta & D \sin n\theta - \sin(n-1)\theta \end{bmatrix} \tag{1}$$

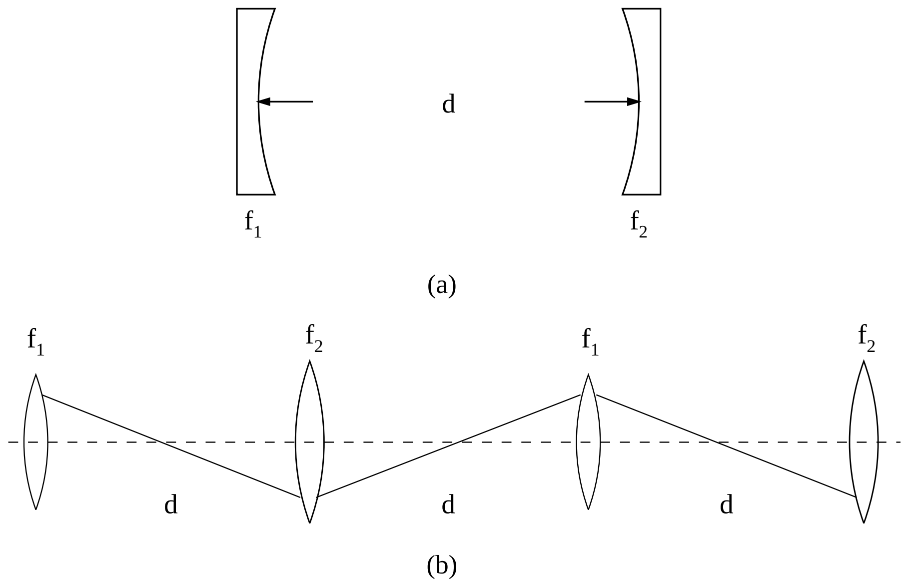


Figure 6: Confocal symmetric resonator and its equivalent lens sequence

where

$$\cos\theta \;=\; \frac{1}{2}(A+D) \qquad \text{and} \qquad \sin\theta = \sqrt{1-\frac{1}{4}(A+D)^2}$$

and clearly,

$$r_{n+1} = ([A\sin n\theta - \sin(n-1)\theta]r_1 + B[\sin n\theta]r_1')/\sin\theta \tag{2}$$

If the beam is supposed to oscillate θ has to be real, else

$$\sin\theta = \sin i\psi = i\sinh\psi$$

and $\sin\theta$ becomes hyperbolic and the ray diverges more and more from the axis as it passes through the system. The condition for θ to be real is

$$-1 < \frac{1}{2}(A+D) < 1$$

We can use this to find out the **stability condition** of the transfer matrix **M**

$$0 \;<\; (1-\frac{d}{2f_1})(1-\frac{d}{2f_2}) \;<\; 1$$

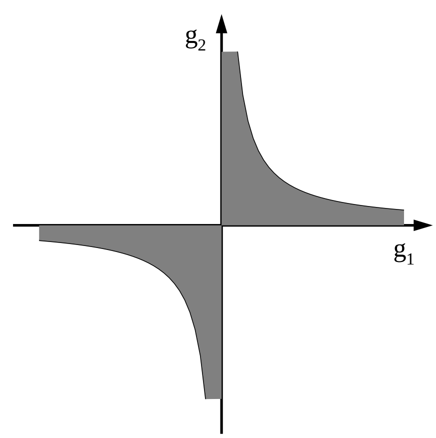


Figure 7: Stability diagram for optical resonator.

By defining

$$g_i = 1-\frac{d}{2f_i} = 1-\frac{d}{R_i}$$

we can write $0 < g_1 g_2 < 1$. This can be shown on a resonator stability diagram, as shown in figure 7.

Fabry-Perot resonator: $R_i = \infty \implies g_1 = g_2 = 1$ Unstable boundary condition

Confocal resonator: $R_i = d \implies g_1 = g_2 = 0$ (un)stable boundary condition

For the confocal resonator the transfer matrix is

$$\mathbf{M}_{conf} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

and therefore

$$\vec{r}_{n+1} = \mathbf{M}_{conf}\vec{r}_n = -\vec{r}_n \qquad \Rightarrow \qquad \vec{r}_{n+2} = \vec{r}_n$$

The confocal resonator is very easy to handle because tilting one of the mirrors is equvalent to

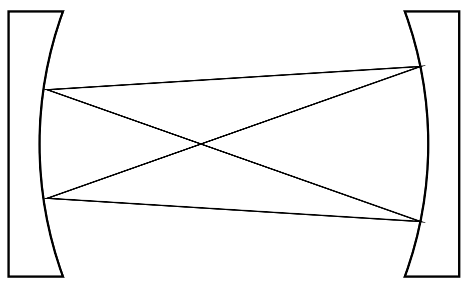


Figure 8: Confocal resonator

move the symmetry axis of the other. The also posess the property that when a laserbeam, that lies outside the axis of symmetry, is directed into the system it will not be reflected back into the laser and disturb it.

3.1.4 Interfaces between two different media

We can see here that $r_2 = r_1$. To solve for the slope we use Snell's law

$$n_1\left(\frac{r_1}{R} - r_1'\right) = n_2\left(\frac{r_1}{R} - r_2'\right) \tag{3}$$

This gives us the transfer matrix

$$\mathbf{M}_i = \begin{bmatrix} 1 & 0 \\ \frac{n_2 - n_1}{n_2}\frac{1}{R} & \frac{n_1}{n_2} \end{bmatrix}$$

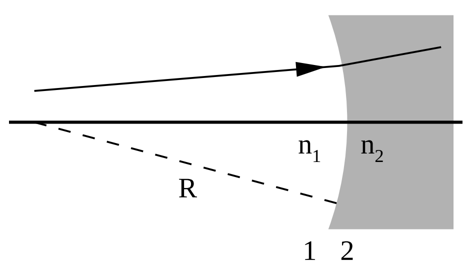


Figure 9: Interface between media