

3 Gaussian beams

The electric field components in a laserbeam must satisfy the scalar wave equation

$$\nabla^2 E - \mu\epsilon\ddot{E} = 0 \quad (1)$$

Suppressing a plane wave factor propagating in the z -direction (the high frequency space and time oscillations) with $E = \psi(\vec{r})e^{-i(kz - \omega t)}$ where $k^2 = \omega^2\mu\epsilon$ we are left with the slowly varying shape function ψ which satisfies the differential equation

$$\nabla_t^2 \psi + \psi'' - 2ik\psi' = 0 \quad (2)$$

The full ∇^2 operator has been split into a transverse part $\nabla_t^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ and an axial part $\frac{\partial^2}{\partial z^2}$ with the double primed shorthand notation. Excluding heavily focused beams, where axial gradients in ψ can be of the same order as the transverse gradients, and sticking to paraxial approximation, we drop the term ψ'' as it is considered negligible in comparison with $2k\psi'$.

The differential equation for the shape factor is now simplified to

$$\nabla_t^2 \psi - 2ik\psi' = 0 \quad (3)$$

We test the strategically shaped trial function

$$\psi = \exp(-i[P(z) + \frac{k}{2q(z)}r^2]) \quad (4)$$

With a complex valued $q(z)$ in the last term of the exponent we generate a Gauss-bell shaped beamprofile and a phasefront curvature. The function $P(z)$ takes care of eventual amplitude and phase development with z . Inserting the trial function into the differential equation and collecting terms of equal power in r , leads to the conditions

$$r^2 \text{ terms} \quad \left(\frac{k}{q}\right)^2 - 2k\frac{kq'}{2q^2} = 0 \quad \implies \quad q' = 1 \quad \implies \quad q = z + iz_0 \quad (5)$$

$$r^0 \text{ terms} \quad \frac{1}{q} - iP'(z) = 0 \quad \implies \quad iP'(z) = \frac{1}{z + iz_0} \quad \implies \quad iP(z) = \ln(1 + \frac{z}{iz_0}) \quad (6)$$

The choice of the integration constants $q(0) = iz_0$ with z_0 real makes the wavefront at $z = 0$ plane, and $P(0) = 0$ means that the phase and amplitude changes are referred to the value at $z = 0$.

The function $q(z)$ goes by the name of q -parameter or beam-parameter, and the characteristic length for the beam z_0 is called Rayleigh length. To separate the real and imaginary parts of the exponent in ψ we introduce two new real variables, R and w , with

$$\frac{1}{q} = \frac{1}{R} - i\frac{1}{\frac{1}{2}kw^2} = \frac{1}{z[1 + (z_0/z)^2]} - i\frac{1}{z_0[1 + (z/z_0)^2]} \quad (7)$$

$$R = z[1 + (z_0/z)^2] \quad \text{and} \quad \frac{1}{2}kw^2 = \frac{1}{2}kw_0^2[1 + (z/z_0)^2] \quad (8)$$

where the Rayleigh length has been written in terms of the minimum value of w ,

$$z_0 = \frac{1}{2}kw_0^2 \quad (9)$$

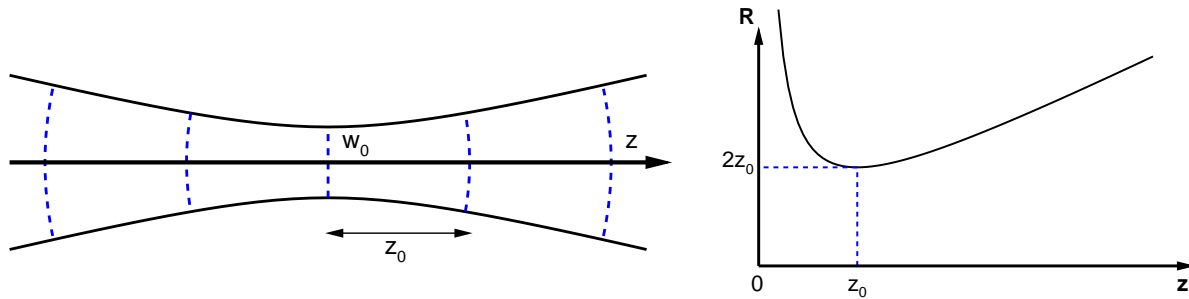
With the new variable w the function iP can also be separated into real and imaginary parts

$$iP(z) = \ln \sqrt{1 + (z/z_0)^2} - i \arctan(z/z_0) = \ln(w/w_0) - i \arctan(z/z_0) \quad (10)$$

In terms of the the new variables the profile function ψ takes the form

$$\psi(z, r) = \frac{w_0}{w} e^{-\frac{r^2}{w^2}} e^{-ik \frac{r^2}{2R}} e^{i \arctan(z/z_0)} \quad (11)$$

The physical meaning of the variable w can be read from the first exponent, as the e^{-1} width of a Gauss-bell shaped profile. The fraction w_0/w ensures correct amplitude scaling as the width of the profile changes. From the second exponent we can interpret R as the radius of curvature of the phasefront. The last exponent describes an extra phase shift in the near field as we pass the beam waist at $z = 0$. The figures below show how the beam width and radius of curvature change with the axial coordinate z .



The radius of curvature takes its minimum value $|R| = 2z_0$ at the Rayleigh length from the waist and the cross-section area of the profile doubles from the waist value, $w^2(z_0) = 2w_0^2$. For $z \gg z_0$ the beam width w approaches $w_0(z/z_0)$ so that the beam divergence halfangle θ is given by

$$\tan \theta = \lim_{z \rightarrow \infty} \frac{dw}{dz} = w_0/z_0 = \frac{\lambda}{\pi w_0} \quad (12)$$

The divergence angle is thus proportional to the wavelength and inversely proportional to the beam waist radius. A narrow beam spreads faster than a wide one.

A He-Ne beam with a field waist radius $w_0 = 0.5$ mm and wavelength $\lambda = 633$ nm has a Rayleigh length $z_0 = \frac{1}{2}kw_0^2 = 124$ cm and a divergence halfangle $\theta = 0.40$ mrad = 0.023° .

The full electrical field spatial distribution and temporal behavior becomes

$$E_{00} = \psi(z, r) e^{-i(kz - \omega t)} = \frac{w_0}{w} e^{-\frac{r^2}{w^2} - ik(z + \frac{r^2}{2R}) + i \arctan(z/z_0) + i\omega t} \quad (13)$$

E_{00} is called the fundamental Gaussian mode. For Cartesian coordinates the higher order transverse modes can be shown to be of the form

$$E_{nm} = H_n \left(\sqrt{2} \frac{x}{w} \right) H_m \left(\sqrt{2} \frac{y}{w} \right) \frac{w_0}{w} e^{-\frac{r^2}{w^2} - ik(z + \frac{r^2}{2R}) + i(n+m+1) \arctan(z/z_0) + i\omega t} \quad (14)$$

where the H_j are Hermite polynomials of degree j . The arctan term in the exponent makes the phase development mode dependent and will separate the transverse modes in frequency inside a resonator. Axial resonator modes are separated with the free spectral length $\Delta\nu = c/2nL$ while separation of the transverse modes depends on the change of $\arctan(z/z_0)$ from one end mirror to the other.

For cylindrical coordinates the higher order transverse modes involve Laguerre polynomials L_p^l for the radial dependence and cosines for the angular coordinate.

3.1 Transformation of Gaussian beams

In geometrical optics we have transformed the radius of curvature of spherical waves with the lens law, although it has mostly been formulated with distances between lens and object/picture. In terms of curvature of spherical waves the lens law states

$$\frac{1}{R_2} = \frac{1}{R_1} - \frac{1}{f} \quad (15)$$

For Gaussian beams we can transform the real part of the beam parameter q in the same way since $\text{Re}(1/q) = 1/R$.

$$\frac{1}{q_2} = \frac{1}{q_1} - \frac{1}{f} \quad (16)$$

This analogy can be taken one step further and take advantage of the full power of the ABCD matrix formalism for more general optical systems. We can identify the ratio r/r' from the matrix language with the spherical phasefront radius of curvature.

$$R = \frac{r}{r'} \quad (17)$$

The matrix equation for the lens transformation can now be written in terms of the R 's as

$$R_2 = \frac{AR_1 + B}{CR_1 + D} \quad (18)$$

In general the q -parameter can be transformed through an arbitrary optical system with the appropriate matrix elements with

$$q_2 = \frac{Aq_1 + B}{Cq_1 + D} \quad (19)$$

Transformation over a distance d in free space is simple; $q_2 = q_1 + d$. The real part of q shows the distance to the waist and the imaginary part the Rayleigh length.

3.2 Matching Gaussian beams to resonators

To get a stable field configuration inside a resonator the beam must reflect onto itself at the end mirrors. The phasefront curvature must therefore match the mirror curvature. Let \mathcal{R} denote mirror radius of curvature and R the phasefront curvature. For resonator mirrors with curvatures \mathcal{R}_1 and \mathcal{R}_2 placed at axial coordinates z_1 and z_2 with centers of curvature placed on the z -axis (alignment requirement) we set up conditions for the beam parameter q

$$\Re(1/q(z_m)) = 1/R(z_m) = 1/\mathcal{R}_m \quad \text{for } m = 1, 2 \quad (20)$$

where $q(z_m) = z_m + iz_0$. The origin of the z -axis is still floating while the spacing of the mirrors $L = z_2 - z_1$ is well defined. We must establish a sign convention for the mirror curvatures \mathcal{R}_m which is in harmony with sign convention for the Gaussian beams; positive when the center of curvature is below the mirror itself on the z -axis (to the left of the mirror on a right pointing axis), otherwise negative.

To identify the resonator beam-mode we must determine the Rayleigh length z_0 and locate the beam waist by finding the mirror position coordinates z_m . We will proceed by using eq. 8 in accordance with the terms set in eq. 20.

$$R_1 = z_1[1 + (z_0/z_1)^2] \quad \text{and} \quad R_2 = z_2[1 + (z_0/z_2)^2] \quad (21)$$

As the graph of $R(z)$ indicates each of these equations have two solutions for z_m , one on each side of $|z| = z_0$. Only the nearfield solution give a physical meaning. Some algebraic acrobatics give the solutions

$$z_1 = \frac{-L(R_2 - L)}{(R_2 - R_1 - 2L)} \quad \text{and} \quad z_2 = \frac{-L(R_1 + L)}{(R_2 - R_1 - 2L)} \quad (22)$$

$$z_0^2 = \frac{-L(R_1 + L)(R_2 - L)(R_2 - R_1 - L)}{(R_2 - R_1 - 2L)^2} \quad (23)$$

Resonator type		z_1	z_2	z_0^2
One plane mirror	$R_1 = \infty$	0	L	$L(R_2 - L)$
Symmetric resonator	$R_1 = -R_2$	$-\frac{L}{2}$	$\frac{L}{2}$	$L(2R_2 - L)$

The waist size is given by

$$w_0^4 = \frac{4z_0^2}{k^2} \quad (24)$$

and the mirror spot sizes by

$$w_1^4 = \frac{4}{k^2} \frac{R_2 - L}{-(R_1 + L)} \frac{LR_1^2}{R_2 - R_1 - L} \quad (25)$$

$$w_2^4 = \frac{4}{k^2} \frac{-(R_1 + L)}{R_2 - L} \frac{LR_2^2}{R_2 - R_1 - L} \quad (26)$$

The resonance conditions can now be written for single passage as

$$\theta(z_2) - \theta(z_1) = p\pi \quad (27)$$

where θ describes the phase development along the axis and p is large integer.

$$\theta(z) = kz - (n + m + 1) \arctan(z/z_0) \quad (28)$$

The cavity resonances then fullfill the condition

$$\nu_{pnm} = \frac{c}{2L} \left[p + (n+m+1) \frac{(\phi_2 - \phi_1)}{\pi} \right] \quad (29)$$

$$\phi_2 = \arctan \left(\sqrt{\frac{-L(R_1 + L)}{(R_2 - L)(R_2 - R_1 - L)}} \right) \quad (30)$$

$$\phi_1 = -\arctan \left(\sqrt{\frac{-L(R_2 - L)}{(R_1 + L)(R_2 - R_1 - L)}} \right) \quad (31)$$

The axial modes (p) are separated by the free spectral length $\Delta\nu = \frac{c}{2L}$ while the transverse modes have a higher density since the factor $(\phi_2 - \phi_1)/\pi$ in eq. 29 is always less than 1/2.

